

# RELATIONS AMONG GAUGE AND PETTIS INTEGRALS FOR MULTIFUNCTIONS WITH WEAKLY COMPACT CONVEX VALUES

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**ABSTRACT.** The aim of this paper is to give some decomposition results for weakly compact and convex valued multifunctions in order to extend a well-known theorem of Fremlin [18, Theorem 8] (a Banach space valued function is McShane integrable if and only if it is Henstock and Pettis integrable) to the multivalued case.

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## 1. INTRODUCTION

A large amount of work about measurable and integrable multifunctions was done in the last decades. Some pioneering and highly influential ideas and notions around the matter were inspired by problems arising in Control Theory and Mathematical Economics. Furthermore this topic is interesting also from the point of view of measure and integration theory, as showed in the papers [1, 2, 6, 9, 10, 15–17, 26]. Inspired by the papers [5, 8, 16, 27], we continue in this paper the study on this subject and we examine relationship among “gauge integrals” (Henstock, Mc Shane, Birkhoff) and Pettis integral of multifunctions whose values are weakly compact and convex subsets of a general Banach space, not necessarily separable.

The name of “gauge integrals” refers to integrals which use in their construction partitions controlled by a positive function, traditionally called a gauge. J. Kurzweil in 1957, and then R. Henstock in 1963, were the first who introduced a definition of a gauge integral for real valued functions, called now the Henstock–Kurzweil integral. Its generalization to vector valued functions or to multivalued functions is called in the literature the Henstock integral. In the family of the gauge integrals there is also the McShane integral and the versions of the Henstock and the McShane integrals when only measurable gauges are allowed ( $\mathcal{H}$  and  $\mathcal{M}$  integrals, respectively), and the variational Henstock and the variational McShane integrals. Moreover according to a recent result of Naralenkov [27, Remark 1], the Birkhoff integral is a gauge integral too. It turned out to be equivalent to the  $\mathcal{M}$  integral.

The main results of our paper are three decomposition theorems (Theorem 3.3, Theorem 4.2 and Theorem 5.2). The first one says that each Henstock integrable multifunction is the sum of a McShane integrable multifunction and of a Henstock

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integrable function. The second one describes each  $\mathcal{H}$ -integrable multifunction as the sum of a Birkhoff integrable multifunction and of an  $\mathcal{H}$ -integrable function and the third one proves that each variationally Henstock integrable multifunction is the sum of a variationally Henstock integrable selection of the multifunction and a Birkhoff integrable multifunction that is also variationally Henstock integrable. As applications of such decomposition results, characterizations of Henstock (Theorem 3.4) and  $\mathcal{H}$  (Theorem 4.3) integrable multifunctions, being extensions of the result given by Fremlin in a remarkable paper [18, Theorem 8] and of more recent results given in [16] and [5], are presented.

## 2. PRELIMINARY FACTS

Throughout the paper  $[0, 1]$  is the unit interval of the real line equipped with the usual topology and Lebesgue measure  $\lambda$ ,  $\mathcal{L}$  denotes the family of all Lebesgue measurable subsets of  $[0, 1]$ , and  $\mathcal{I}$  is the collection of all closed subintervals of  $[0, 1]$ : if  $I \in \mathcal{I}$  then its Lebesgue measure will be denoted by  $|I|$ .

A finite partition  $\mathcal{P}$  in  $[0, 1]$  is a collection  $\{(I_1, t_1), \dots, (I_p, t_p)\}$ , where  $I_1, \dots, I_p$  are nonoverlapping (i.e. the intersection of two intervals is at most a singleton) subintervals of  $[0, 1]$ ,  $t_i$  is a point of  $[0, 1]$ ,  $i = 1, \dots, p$ .

If  $\cup_{i=1}^p I_i = [0, 1]$ , then  $\mathcal{P}$  is a *partition* of  $[0, 1]$ . If  $t_i$  is a point of  $I_i$ ,  $i = 1, \dots, p$ , we say that  $\mathcal{P}$  is a *Perron partition* of  $[0, 1]$ .

A countable partition  $(A_n)$  of  $[0, 1]$  in  $\mathcal{L}$  is a collections of pairwise disjoint  $\mathcal{L}$ -measurable sets such that  $\cup_n A_n = [0, 1]$ . We admit empty sets.

A *gauge* on  $[0, 1]$  is any positive function on  $[0, 1]$ . For a given gauge  $\delta$  on  $[0, 1]$ , we say that a partition  $\{(I_1, t_1), \dots, (I_p, t_p)\}$  is  $\delta$ -*fine* if  $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ ,  $i = 1, \dots, p$ .

$X$  is an arbitrary Banach space with its dual  $X^*$ . The closed unit ball of  $X^*$  is denoted by  $B_{X^*}$ .  $cwk(X)$  is the family of all non-empty convex weakly compact subsets of  $X$ . We consider on  $cwk(X)$  the Minkowski addition ( $A+B := \{a+b : a \in A, b \in B\}$ ) and the standard multiplication by scalars.  $\|A\| := \sup\{\|x\| : x \in A\}$ .  $d_H$  is the Hausdorff distance in  $cwk(X)$ . The space  $cwk(X)$  with the Hausdorff distance is a complete metric space. For every  $C \in cwk(X)$  the *support function* of  $C$  is denoted by  $s(\cdot, C)$  and defined on  $X^*$  by  $\sigma(x^*, C) := \sup\{\langle x^*, x \rangle : x \in C\}$ , for each  $x^* \in X^*$ .

**Definition 2.1.** A map  $\Gamma : [0, 1] \rightarrow cwk(X)$  is called a *multifunction*. A map  $\Gamma : \mathcal{I} \rightarrow cwk(X)$  is called an *interval multifunction*. A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *scalarly measurable* if for every  $x^* \in X^*$ , the map  $\sigma(x^*, \Gamma(\cdot))$  is measurable.

$\Gamma$  is said to be *Bochner measurable* if there exists a sequence of simple multifunctions  $\Gamma_n : [0, 1] \rightarrow cwk(X)$  such that  $\lim_{n \rightarrow \infty} d_H(\Gamma_n(t), \Gamma(t)) = 0$  for almost all  $t \in [0, 1]$ . It is well known that Bochner measurability of a  $cwk(X)$ -valued multifunction yields its scalar measurability (if  $X$  is separable also the reverse implication is true). If a multifunction is a function, then we use the traditional name of strong measurability. A function  $f : [0, 1] \rightarrow X$  is called a *selection* of  $\Gamma$  if  $f(t) \in \Gamma(t)$ , for every  $t \in [0, 1]$ .

**Definition 2.2.** [9, Proposition 2.6] A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *Birkhoff integrable* on  $[0, 1]$ , if there exists a set  $\Phi_\Gamma([0, 1]) \in cwk(X)$  with the following property: for every  $\varepsilon > 0$  there is a countable partition  $\Pi_0$  of  $[0, 1]$

in  $\mathcal{L}$  such that for every countable partition  $\Pi = (A_n)_n$  of  $[0, 1]$  in  $\mathcal{L}$  finer than  $\Pi_0$  and any choice  $T = (t_n)_n$  in  $A_n$ , the series  $\sum_n \lambda(A_n) \Gamma(t_n)$  is unconditionally convergent (in the sense of the Hausdorff metric) and

$$(1) \quad d_H(\Phi_\Gamma([0, 1]), \sum_n \Gamma(t_n) \lambda(A_n)) < \varepsilon.$$

**Definition 2.3.** A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *Henstock* (resp. *McShane*) *integrable* on  $[0, 1]$ , if there exists  $\Phi_\Gamma([0, 1]) \in cwk(X)$  with the property that for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[0, 1]$  such that for each  $\delta$ -fine Perron partition (resp. partition)  $\{(I_1, t_1), \dots, (I_p, t_p)\}$  of  $[0, 1]$ , we have

$$(2) \quad d_H(\Phi_\Gamma([0, 1]), \sum_{i=1}^p \Gamma(t_i) |I_i|) < \varepsilon.$$

If the gauges above are taken to be measurable, then we speak of  $\mathcal{H}$  (resp.  $\mathcal{M}$ )-integrability on  $[0, 1]$ . A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *Henstock* (resp.  $\mathcal{H}$ ) *integrable* on  $I \in \mathcal{I}$  if  $\Gamma 1_I$  is Henstock (resp.  $\mathcal{H}$ ) integrable on  $[0, 1]$ .

In case the multifunction is a single valued function and  $X$  is the real line, the corresponding integral is called *Henstock-Kurzweil integral* (or *HK-integral*) and it is denoted by the symbol  $(HK) \int_I$ .

$\Gamma$  is said to be *McShane* (resp.  $\mathcal{M}$ ) *integrable* on  $E \in \mathcal{L}$  if  $\Gamma 1_E$  is integrable on  $[0, 1]$  in the corresponding sense. We write then  $(H) \int_I \Gamma dt := \Phi_{\Gamma 1_I}([0, 1])$  (resp.  $(\mathcal{H}) \int_I \Gamma dt := \Phi_{\Gamma 1_I}([0, 1])$ ),  $(MS) \int_E \Gamma dt := \Phi_{\Gamma 1_E}([0, 1])$  or  $(\mathcal{M}) \int_E \Gamma dt := \Phi_{\Gamma 1_E}([0, 1])$ . It is known that a multifunction that is Henstock ( $\mathcal{H}$ ) integrable on  $[0, 1]$  is in the same manner integrable on each  $I \in \mathcal{I}$  and if  $\Gamma : [0, 1] \rightarrow cwk(X)$  is McShane ( $\mathcal{M}$ ) integrable on  $[0, 1]$ , then it is in the same manner integrable on every  $E \in \mathcal{L}$  (see e.g. [16]).

**Definition 2.4.** [14, Definition 2] A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *Henstock-Kurzweil-Pettis integrable* (or *HKP-integrable*) on  $[0, 1]$  if for every  $x^* \in X^*$  the map  $\sigma(x^*, \Gamma(\cdot))$  is HK-integrable and for each  $I \in \mathcal{I}$  there exists a set  $W_I \in cwk(X)$  such that  $\sigma(x^*, W_I) = (HK) \int_I \sigma(x^*, \Gamma)$ , for every  $x^* \in X^*$ . The set  $W_I$  is called the *Henstock-Kurzweil-Pettis integral* of  $\Gamma$  over  $I$  and we set  $W_I := (HKP) \int_I \Gamma$ .

For the definitions and properties of Pettis and of Henstock-Kurzweil-Pettis integral for functions and multifunctions we refer the reader to [10, 15, 16, 20, 23–26].

An interval multifunction  $\Phi : \mathcal{I} \rightarrow cwk(X)$  is said to be *finitely additive*, if for every non-overlapping intervals  $I_1, I_2 \in \mathcal{I}$  such that  $I_1 \cup I_2 \in \mathcal{I}$  we have  $\Phi(I_1 \cup I_2) = \Phi(I_1) + \Phi(I_2)$ . In this case  $\Phi$  is said to be an *interval multimeasure*. A map  $M : \mathcal{L} \rightarrow cwk(X)$  is said to be a  $d_H$ -*multimeasure* if for every sequence  $(A_n)_{n \geq 1}$  in  $\mathcal{L}$  of pairwise disjoint sets with  $A = \bigcup_{n \geq 1} A_n$ , we have  $d_H(M(A), \sum_{k=1}^n M(A_k)) \rightarrow 0$ , as  $n \rightarrow +\infty$ .

**Remark 2.5.** It is well known that  $M$  is a  $d_H$ -multimeasure if and only if it is a multimeasure, i.e. if for every  $x^* \in X^*$ , the map  $\mathcal{L} \ni A \mapsto s(x^*, M(A))$  is a real valued measure (see [21, Theorem 8.4.10]). Observe moreover that this is a multivalued analogue of Orlicz-Pettis theorem).

We say that the multimeasure  $M : \mathcal{L} \rightarrow cwk(X)$  is  $\lambda$ -continuous and we write  $M \ll \lambda$ , if for every  $A \in \mathcal{L}$  such that  $\lambda(A) = 0$  yields  $M(A) = \{0\}$ .

It is known that the primitives of Henstock or  $\mathcal{H}$  integrable multifunctions are interval multimeasures, while the primitives of McShane or Birkhoff integrable multifunctions are multimeasures.

**Definition 2.6.** A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is said to be *variationally Henstock (variationally McShane) integrable*, if there exists a finitely additive interval multifunction  $\Phi_\Gamma : \mathcal{I} \rightarrow cwk(X)$  with the following property: for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $[0, 1]$  such that for each  $\delta$ -fine Perron partition (partition)  $\{(I_1, t_1), \dots, (I_p, t_p)\}$  of  $[0, 1]$ , we have

$$(3) \quad \sum_{j=1}^p d_H(\Phi_\Gamma(I_j), \Gamma(t_j)|I_j|) < \varepsilon.$$

We write then  $(vH) \int_0^1 \Gamma dt := \Phi_\Gamma([0, 1])$  ( $(vMS) \int_0^1 \Gamma dt := \Phi_\Gamma([0, 1])$ ). We call the set multifunction  $\Phi_\Gamma$  the *variational Henstock (McShane) primitive* of  $\Gamma$ . The variational integrals on  $I \in \mathcal{I}$  are defined in the same way as the ordinary ones. All the integrals defined as far are uniquely determined.

For the definitions of Pettis and of Henstock-Kurzweil-Pettis integral for functions and multifunctions we refer the reader to [10, 15, 16, 23–26].

A useful tool to study the integrability of a single-valued function or of a multifunction is the variational measure associated to the primitive.

**Definition 2.7.** Given an interval multimeasure  $\Phi : \mathcal{I} \rightarrow cwk(X)$ , a gauge  $\delta$  and a set  $E \subset [0, 1]$ , we define

$$Var(\Phi, \delta, E) = \sup \sum_{j=1}^p \|\Phi(I_j)\|,$$

where the supremum is taken over all the  $\delta$ -fine Perron partitions  $\{(I_j, t_j)\}_{j=1}^p$  with  $t_j \in E$  for  $j = 1, \dots, p$ . The set function

$$V_\Phi(E) := \inf_{\delta} \{Var(\Phi, \delta, E) : \delta \text{ is a gauge on } E\}$$

is called *the variational measure generated by  $\Phi$* . Moreover,  $V_\Phi \ll \lambda$  if for every  $E \in \mathcal{L}$  with  $\lambda(E) = 0$  we have  $V_\Phi(E) = 0$ .

For its properties we refer the reader to [4] or [17].

Another important technique to study the multifunctions are the embeddings, see for example [22]. Let  $l_\infty(B(X^*))$  be the Banach space of bounded real valued functions defined on  $B(X^*)$  endowed with the supremum norm  $\|\cdot\|_\infty$ . The Rådström embedding  $i : cwk(X) \rightarrow l_\infty(B(X^*))$  given by  $i(A)(x^*) := \sigma(x^*, A)$  satisfies the following properties:

- 1)  $i(\alpha A + \beta C) = \alpha i(A) + \beta i(C)$  for every  $A, C \in cwk(X)$ ,  $\alpha, \beta \in \mathbb{R}^+$ ;
- 2)  $d_H(A, C) = \|i(A) - i(C)\|_\infty$ ,  $A, C \in cwk(X)$ ;
- 3)  $i(cwk(X)) = \overline{i(cwk(X))}$ .

It follows directly from the definitions that if  $i$  is the Rådström embedding into  $l_\infty(B(X^*))$ , then a multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is  $G$ -integrable if and only if  $i \circ \Gamma$  is  $G$ -integrable ( $G$  stands for any of the gauge integrals).

We also recall that, for a Pettis integrable mapping  $G : [0, 1] \rightarrow cwk(X)$ , its integral  $J_G$  is a multimeasure on the  $\sigma$ -algebra  $\mathcal{L}$  (cf. [11, Theorem 4.1]) that is  $\lambda$ -continuous. As also observed in [11, section 3], this means that the *embedded* measure  $i(J_G)$  is a countably additive measure with values in  $l_\infty(B(X^*))$ .

In case of single valued functions (studied also in [4, 7, 13]), according to a result of Naralakov [27, Remark 1],  $\mathcal{M}$ -integrability is equivalent to the Birkhoff integrability. Thanks to the embedding, the equivalence is also true for multifunctions. So we will use the traditional name of Birkhoff integral instead of  $\mathcal{M}$ -integral.

We recall that

**Definition 2.8.** [27, Definition 2] A function  $f : [0, 1] \rightarrow X$  is said to be *Riemann measurable* on  $[0, 1]$  if for every  $\varepsilon > 0$  there exist a  $\delta > 0$  and a closed set  $F \subset [0, 1]$  with  $\lambda([0, 1] \setminus F) < \varepsilon$  exist such that

$$\left\| \sum_{i=1}^p \{f(t_i) - f(t'_i)\} |I_i| \right\| < \varepsilon$$

whenever  $\{I_i\}$  is a finite collection of pairwise nonoverlapping intervals with  $\max_{1 \leq i \leq p} |I_i| < \delta$  and  $t_i, t'_i \in I_i \cap F$ .

According to [27, Theorem 4] each  $\mathcal{H}$ -integrable function is Riemann measurable on  $[0, 1]$ . Moreover in [8, Theorem 9] it was proved that a function  $f : [0, 1] \rightarrow X$  is  $\mathcal{M}$ -integrable if and only if  $f$  is both Riemann measurable and Pettis integrable. So we get the following characterization, that is parallel to Fremlin's description [18]:

**Theorem 2.9.** *A function  $f : [0, 1] \rightarrow X$  is Birkhoff integrable if and only if it is  $\mathcal{H}$ -integrable and Pettis integrable.*

*Proof.* The if part is trivial. For the converse observe that  $\mathcal{H}$ -integrability implies Riemann measurability by [27, Theorem 4]. Moreover by [18, Theorem 8]  $f$  is Mc Shane integrable and Riemann measurability together with Mc Shane integrability implies  $\mathcal{M}$ -integrability by [27, Theorem 7].  $\square$

We denote by  $\mathcal{S}_P(\Gamma), \mathcal{S}_{MS}(\Gamma), \mathcal{S}_{\mathcal{H}}(\Gamma), \mathcal{S}_{\mathcal{M}}(\Gamma), \mathcal{S}_H(\Gamma), \mathcal{S}_{Bi}(\Gamma), \mathcal{S}_{vH}(\Gamma)$ , the collections of all selections of  $\Gamma : [0, 1] \rightarrow cwk(X)$  which are respectively Pettis, Mc Shane,  $\mathcal{H}$ ,  $\mathcal{M}$ , Henstock, Birkhoff and variationally Henstock integrable.

### 3. HENSTOCK AND MCSHANE INTEGRABILITY OF $cwk(X)$ -VALUED MULTIFUNCTIONS

**Proposition 3.1.** [17, Proposition 4.4] *Let  $\Phi : \mathcal{I} \rightarrow cwk(X)$  be an interval multimeasure such that  $V_{\sigma(x^*, \Phi)} \ll \lambda$  for every  $x^* \in X^*$ . Assume that  $\sigma(x^*, \Phi(I)) \geq 0$  for every  $x^* \in X^*$  and for every  $I \in \mathcal{I}$ . Then  $\Phi$  can be extended to a multimeasure  $M : \mathcal{L} \rightarrow cwk(X)$  with  $M \ll \lambda$ .*

*Proof.* In the statement of [17, Proposition 4.4] the hypothesis  $V_{\sigma(x^*, \Phi)} \ll \lambda$  was substituted by the stronger condition  $V_\Phi \ll \lambda$ , but in the proof the  $V_{\sigma(x^*, \Phi)} \ll \lambda$  is the condition used.

□

**Proposition 3.2.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be such that  $\Gamma(\cdot) \ni 0$  a.e. If  $\Gamma$  is Henstock integrable (resp.  $\mathcal{H}$ -integrable) on  $[0, 1]$ , then it is also McShane (resp. Birkhoff, i.e.  $\mathcal{M}$ ) integrable on  $[0, 1]$ .*

*Proof.* Let  $i$  be the Rådström embedding of  $cwk(X)$  into  $l_\infty(B_{X^*})$ . If  $\Gamma$  is Henstock integrable, then we just have to prove that  $i \circ \Gamma$  is McShane integrable. By the hypothesis we have that  $i \circ \Gamma$  is Henstock integrable. Then, thanks to [18, Corollary 9 (iii)], it will be sufficient to prove convergence in  $l_\infty(B(X^*))$  of all series of the type  $\sum_n (H) \int_{I_n} i \circ \Gamma$ , where  $(I_n)_n$  is any sequence of pairwise non-overlapping subintervals of  $[0, 1]$ .

Let  $\mathcal{A}$  be the ring generated by the subintervals  $[a, b] \subseteq [0, 1]$ . The map  $\Phi(E) := (H) \int_E \Gamma$  is well defined and finitely additive on  $\mathcal{A}$ . Since  $0 \in \Gamma(t)$ , we have  $\sigma(x^*, \Phi(E)) \geq 0$  for every  $x^* \in X^*$  and every  $E \in \mathcal{A}$ . Since the scalar function  $\sigma(x^*, \Gamma)$  is non negative, it is also Lebesgue integrable. By [4, Lemma 3.3] the variational measure of its integral measure is  $\lambda$ -continuous (i.e.  $V_{\sigma(x^*, \Phi)} \ll \lambda$ ). Thus, by Proposition 3.1 the map  $\Phi$  can be extended to a  $\lambda$ -continuous multimeasure  $\tilde{\Phi} : \mathcal{L} \rightarrow cwk(X)$ . So, fixed any sequence of pairwise non-overlapping subintervals  $(I_n)$  of  $[0, 1]$ , let  $E_k = \cup_{n=1}^k I_n$   $k \in \mathbb{N}$ . Then

$$i(\tilde{\Phi})(E_k) := \sum_{n=1}^k (H) \int_{I_n} i \circ \Gamma \in l_\infty(B(X^*))$$

and thanks to the countable additivity of  $i(\tilde{\Phi})$  the series  $\sum_{n=1}^\infty (H) \int_{I_n} i \circ \Gamma$  converges in  $l_\infty(B(X^*))$ . As said before, thanks to [18, Corollary 9 (iii)],  $i \circ \Gamma$  is McShane integrable. Consequently,  $\Gamma$  is McShane integrable.

If  $\Gamma$  is  $\mathcal{H}$ -integrable, then  $i \circ \Gamma$  is  $\mathcal{H}$ -integrable and being already McShane integrable, it is also Pettis integrable [18, Theorem 8]. Applying now Theorem 2.9, we obtain Birkhoff integrability of  $i \circ \Gamma$ . This yields Birkhoff integrability of  $\Gamma$ . □

We recall that a function  $f : [0, 1] \rightarrow X$  is said to be scalarly *Henstock-Kurzweil integrable* if for every  $x^* \in X^*$  the real function  $x^*f(\cdot)$  is Henstock-Kurzweil integrable.

**Theorem 3.3.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a scalarly Kurzweil-Henstock integrable multifunction. Then the following conditions are equivalent:*

- (i)  $\Gamma$  is Henstock integrable;
- (ii)  $\mathcal{S}_H(\Gamma) \neq \emptyset$  and for every  $f \in \mathcal{S}_H(\Gamma)$  the multifunction  $\Gamma - f$  is McShane integrable;
- (iii) there exists  $f \in \mathcal{S}_H(\Gamma)$  such that the multifunction  $G := \Gamma - f$  is McShane integrable.

*Proof.* **(i)  $\Rightarrow$  (ii)** According to [16, Theorem 3.1]  $\mathcal{S}_H(\Gamma) \neq \emptyset$ . Let  $f \in \mathcal{S}_H(\Gamma)$  be fixed. Then  $\Gamma - f$  is also Henstock integrable (in  $cwk(X)$ ) and  $0 \in \Gamma - f$  for every  $t \in [0, 1]$ . By Proposition 3.2 the multifunction  $\Gamma - f$  is McShane integrable. Since each McShane integrable multifunction is also Henstock integrable, **(ii)  $\Rightarrow$  (iii)** is trivial, **(iii)  $\Rightarrow$  (i)** follows at once. □

The next result generalizes [16, Theorem 3.4], proved there for  $cwk(X)$ -valued multifunctions with compact valued integrals.

**Theorem 3.4.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a scalarly measurable multifunction. Then the following conditions are equivalent:*

- (i)  $\Gamma$  is McShane integrable;
- (ii)  $\Gamma$  is Henstock integrable and  $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$ .
- (iii)  $\Gamma$  is Henstock integrable and  $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_P(\Gamma)$ ;
- (iv)  $\Gamma$  is Henstock integrable and  $\mathcal{S}_P(\Gamma) \neq \emptyset$ .
- (v)  $\Gamma$  is Henstock and Pettis integrable.

*Proof.* (i)  $\Rightarrow$  (ii) Pick  $f \in \mathcal{S}_H(\Gamma)$  then, according to Theorem 3.3,  $\Gamma = G + f$  for a McShane integrable  $G$ . But as  $\Gamma$  is Pettis integrable, also  $f$  is Pettis integrable (cf. [25, Corollary 1.5]). In view of [18, Theorem 8],  $f$  is McShane integrable. (ii)  $\Rightarrow$  (iii) is valid, because each McShane integrable function is also Pettis integrable ([19, Theorem 2C]). (iii)  $\Rightarrow$  (iv) In view of [16, Theorem 3.1]  $\mathcal{S}_H(\Gamma) \neq \emptyset$  and so (iii) implies  $\mathcal{S}_P(\Gamma) \neq \emptyset$ .

(iv)  $\Rightarrow$  (v) Take  $f \in \mathcal{S}_P(\Gamma)$ . Since  $\Gamma$  is Henstock integrable, it is also HKP-integrable and so applying [15, Theorem 2] we obtain a representation  $\Gamma = G + f$ , where  $G : [0, 1] \rightarrow cwk(X)$  is Pettis integrable in  $cwk(X)$ . Consequently,  $\Gamma$  is also Pettis integrable in  $cwk(X)$  and so (v) holds.

(v)  $\Rightarrow$  (i) In virtue of [16, Theorem 3.1]  $\Gamma$  has a McShane integrable selection  $f$ . It follows from Theorem 3.3 that the multifunction  $G : [0, 1] \rightarrow cwk(X)$  defined by  $\Gamma(t) = G(t) + f(t)$  is McShane integrable.  $\square$

#### 4. BIRKHOFF AND $\mathcal{H}$ -INTEGRABILITY OF $cwk(X)$ -VALUED MULTIFUNCTIONS

A quick analysis of the proof of [16, Theorem 3.1] proves the following:

**Proposition 4.1.** *If  $\Gamma : [0, 1] \rightarrow cwk(X)$  is  $\mathcal{H}$ -integrable, then  $\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$ . If  $\Gamma : [0, 1] \rightarrow cwk(X)$  is Pettis and  $\mathcal{H}$ -integrable, then  $\mathcal{S}_{Bi}(\Gamma) \neq \emptyset$ .*

As a consequence, we have the following result:

**Theorem 4.2.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a scalarly Kurzweil–Henstock integrable multifunction. Then the following conditions are equivalent:*

- (i)  $\Gamma$  is  $\mathcal{H}$ -integrable;
- (ii)  $\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$  and for every  $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$  the multifunction  $\Gamma - f$  is Birkhoff integrable;
- (iii) there exists  $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$  such that the multifunction  $\Gamma - f$  is Birkhoff integrable.

*Proof.* (i)  $\Rightarrow$  (ii) Instead of [16, Theorem 3.1] we apply Proposition 4.1.  $\square$

Applying Theorems 4.2 and 2.9, we have the following:

**Theorem 4.3.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a scalarly measurable multifunction. Then the following conditions are equivalent:*

- (i)  $\Gamma$  is Birkhoff integrable;
- (ii)  $\Gamma$  is  $\mathcal{H}$ -integrable and  $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_{Bi}(\Gamma)$ .
- (iii)  $\Gamma$  is  $\mathcal{H}$ -integrable and  $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$ .
- (iv)  $\Gamma$  is  $\mathcal{H}$ -integrable and  $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_P(\Gamma)$ ;
- (v)  $\Gamma$  is  $\mathcal{H}$ -integrable and  $\mathcal{S}_P(\Gamma) \neq \emptyset$ .
- (vi)  $\Gamma$  is Pettis and  $\mathcal{H}$ -integrable.



*Proof.* (i)  $\Rightarrow$  (ii) If  $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$  then, according to Theorem 4.2,  $\Gamma = G + f$  for a Birkhoff integrable  $G$ . But as  $\Gamma$  is Pettis integrable, also  $f$  is Pettis integrable (cf. [25, Corollary 1.5]). In view of Theorem 2.9  $f$  is Birkhoff integrable.

(ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) are valid, because each Birkhoff integrable function is McShane integrable and each McShane integrable function is also Pettis integrable ([19, Theorem 2C]). (iv)  $\Rightarrow$  (v) In view of Proposition 4.1  $\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$  and so (iii) implies  $\mathcal{S}_P(\Gamma) \neq \emptyset$ .

(v)  $\Rightarrow$  (vi) Take  $f \in \mathcal{S}_P(\Gamma)$ . Since  $\Gamma$  is  $\mathcal{H}$ -integrable, it is also HKP-integrable and so applying [15, Theorem 2] we obtain a representation  $\Gamma = G + f$ , where  $G : [0, 1] \rightarrow cwk(X)$  is Pettis integrable in  $cwk(X)$ . Consequently,  $\Gamma$  is also Pettis integrable in  $cwk(X)$  and so (v) holds.

(vi)  $\Rightarrow$  (i) In virtue of Proposition 4.1  $\Gamma$  has a Birkhoff integrable selection  $f$ . It follows from Theorem 4.2 that the multifunction  $G : [0, 1] \rightarrow cwk(X)$  defined by  $G := \Gamma - f$  is Birkhoff integrable.  $\square$

## 5. VARIATIONALLY HENSTOCK INTEGRABLE SELECTIONS

Now we are going to consider the existence of variationally Henstock integrable selections for a variationally Henstock integrable multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$ . As proved in [5, Theorem 3.12], if  $X$  has the Radon-Nikodým property, then such mappings have variationally Henstock integrable selections.

We shall now prove the following result.

**Theorem 5.1.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be any variationally Henstock integrable multifunction. Then every strongly measurable selection of  $\Gamma$  is also variationally Henstock integrable.*

*Proof.* Let  $f$  be a strongly measurable selection of  $\Gamma$ . Then  $f$  is Henstock-Kurtzweil-Pettis integrable and the mapping  $G$  defined by  $G := \Gamma - f$  is Pettis integrable: see [15, Theorem 1]. Since  $\Gamma$  is vH-integrable then  $\Gamma$  is Bochner measurable ([5, Proposition 2.8]). As the difference of  $i(\Gamma)$  and  $i(\{f\})$ ,  $i(G)$  is strongly measurable, together with  $G$ . Therefore  $G$  has essentially  $d_H$ -separable range (that is, there is  $E \in \mathcal{L}$ , with  $\lambda([0, 1] \setminus E) = 0$  and  $G(E)$  is  $d_H$ -separable) and  $i(G)$  is also Pettis integrable (see [9, Theorem 3.4 and Lemma 3.3 and their proofs] and [10, comments after Theorem A]).

Now, since  $\Gamma$  is variationally Henstock integrable, the variational measure  $V_{\Phi}$  associated to the vH-integral of  $\Gamma$  is absolutely continuous (see [28, Proposition 3.3.1]). If  $V_{\phi}$  is associated to the Henstock-Kurzweil-Pettis integral of  $f$ , then  $V_{\phi} \leq V_{\Phi}$  and so it is also absolutely continuous with respect to  $\lambda$ . Since  $\|G\| \leq \|\Gamma\| + \|f\|$ , it is clear that also  $V_G$  is  $\lambda$ -continuous.

Then,  $i(G)$  satisfies all the hypotheses of [4, Corollary 4.1] and therefore it is variationally Henstock integrable. But then  $i(\{f\})$  is too, as the difference of  $i(\Gamma)$  and  $i(G)$ , and finally  $f$  is variationally Henstock integrable.  $\square$

Thus, we infer immediately:

**Corollary 5.1.** *If  $\Gamma : [0, 1] \rightarrow cwk(X)$  is a variationally Henstock integrable multifunction, then  $\Gamma$  has a variationally Henstock integrable selection.*

*Proof.* Since  $\Gamma$  possess strongly measurable selections [5, Proposition 3.3], the thesis follows at once by Theorem 5.1.  $\square$



**Remark 5.2.** At this point it is worth to observe that the thesis of Theorem 5.1 holds true only for strongly measurable selections of  $\Gamma$ . In general  $\Gamma$  may have scalarly measurable selections which are neither strongly measurable nor even Henstock integrable (see [5, Proposition 3.2]).

A decomposition result, similar to Theorem 4.2, can be formulated now

**Theorem 5.2.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a variationally Henstock integrable multifunction. Then  $\Gamma$  is the sum of a variationally Henstock integrable selection  $f$  and a Birkhoff integrable multifunction  $G : [0, 1] \rightarrow cwk(X)$  that is variationally Henstock integrable.*

*Proof.* Let  $f$  be any variationally Henstock integrable selection of  $\Gamma$ . Then, as previously proved,  $\Gamma$  is Bochner measurable,  $f$  is strongly measurable and the variational measures associated with their integral functions are  $\lambda$ -continuous. Moreover,  $f$  is HKP-integrable and, according to [15, Theorem 1], the multifunction  $G$ , defined by  $G := \Gamma - f$ , is Pettis integrable. Since  $\Gamma$  and  $f$  are variationally Henstock integrable the same holds true for  $G$ . Hence also  $i(G)$  is variationally Henstock integrable and, consequently, by [5, Proposition 4.1],  $G$  is also Birkhoff integrable.  $\square$

**Remark 5.3.** There is now an obvious question: Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a variationally Henstock integrable multifunction. Does there exist a variationally Henstock integrable selection  $f$  of  $\Gamma$  such that  $G := \Gamma - f$  is variationally McShane integrable? Unfortunately, in general, the answer is negative. The argument is similar to that applied in [14]. Assume that  $X$  is separable and  $f$  is an  $X$ -valued function that is vH but not vMS-integrable. Let  $\Gamma(t) := \text{conv}\{0, f(t)\}$ . Then  $\Gamma$  is vH-integrable but according to [5, Theorem 3.7] it is not vMS-integrable and possesses at least one strongly measurable selection  $g$  that is not Bochner integrable. But by Theorem 5.1  $g$  is vH-integrable. Consider the multifunction  $G = \Gamma - g$ . Clearly  $G$  is vH-integrable. But  $G(t) = \text{conv}\{-g(t), f(t) - g(t)\}$  for all  $t \in [0, 1]$  and so  $-g$  is a selection of  $G$  that is not Bochner integrable. It follows from [5, Theorem 3.7] that  $G$  is not variationally McShane integrable.

The next two theorems have the same proofs as the corresponding results in [5].

**Theorem 5.4.** *Let  $\Gamma : [0, 1] \rightarrow cwk(X)$  be a vH-integrable multifunction. Then the following conditions are equivalent:*

- (a)  $\mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$ ;
- (b)  $\mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_P(\Gamma)$ ;
- (c)  $\mathcal{S}_P(\Gamma) \neq \emptyset$ ;
- (d)  $\Gamma$  is Pettis integrable.
- (e)  $\Gamma$  is McShane integrable.

We proved in [5, Proposition 3.6] that a Bochner measurable multifunction is variationally McShane integrable if and only if it is integrably bounded. Since each variationally Henstock integrable multifunction is Bochner measurable, we obtain the following result:

**Theorem 5.5.** *A multifunction  $\Gamma : [0, 1] \rightarrow cwk(X)$  is variationally McShane integrable if and only if it is variationally Henstock integrable and integrably bounded. In particular, if  $\Gamma : [0, 1] \rightarrow cwk(X)$  is variationally Henstock integrable and integrably bounded, then all the statements given in Theorem 3.4 are equivalent to variational McShane integrability of  $\Gamma$ .*

**Corollary 5.6.** *A function  $f : [0, 1] \rightarrow X$  is variationally McShane integrable (= Bochner integrable, cf. [13]) if and only if it is variationally Henstock integrable and integrably bounded.*

## 6. VARIATIONAL $\mathcal{H}$ -INTEGRAL

Recently, Naralnikov introduced stronger forms of Henstock and McShane integrals of functions, and called them  $\mathcal{H}$  and  $\mathcal{M}$  integrals. We apply that idea to variational integrals. Since the variational McShane integral of functions coincides with Bochner integral, the same holds true for the  $\mathcal{M}$ -integral. In case of the variational  $\mathcal{H}$ -integral the situation is not as obvious, but we shall prove in this section that the variational  $\mathcal{H}$ -integral coincides with the variational Henstock integral. We begin with the following strengthening of the Riemann measurability, due to [27].

**Definition 6.1.** We say that a function  $f : [0, 1] \rightarrow X$  is *strongly Riemann measurable*, if for every  $\varepsilon > 0$  there exist a positive number  $\delta$  and a closed set  $F \subset [0, 1]$  such that  $\lambda([0, 1] \setminus F) < \varepsilon$  and

$$(4) \quad \sum_{k=1}^K \|f(t_k) - f(t'_k)\| |I_k| < \varepsilon$$

whenever  $\{I_1, \dots, I_K\}$  is a nonoverlapping finite family of subintervals of  $[0, 1]$  with  $\max_k |I_k| < \delta$  and, all points  $t_k, t'_k$  are chosen in  $I_k \cap F$ ,  $k = 1, \dots, K$ .

**Lemma 6.2.** *If  $f : [0, 1] \rightarrow X$  is strongly measurable, then  $f$  is strongly Riemann measurable.*

*Proof.* Fix  $\varepsilon > 0$ . Then there exists a closed set  $F \subset [0, 1]$  such that  $\lambda([0, 1] \setminus F) < \varepsilon$  and  $f|_F$  is continuous. Since  $F$  is compact, then  $f|_F$  is uniformly continuous, and so there exists a positive number  $\delta > 0$  such that, as soon as  $t, t'$  are chosen in  $F$ , with  $|t - t'| < \delta$ , then  $\|f(t) - f(t')\| < \varepsilon$ . Now, fix any finite family  $\{I_1, \dots, I_K\}$  of non-overlapping intervals with  $\max_k |I_k| < \delta$ , and choose arbitrarily points  $t_k, t'_k$  in  $I_k \cap F$  for every  $k$ : then we have

$$\sum_{k=1}^K \|f(t_k) - f(t'_k)\| |I_k| < \sum_{k=1}^K \varepsilon |I_k| < \varepsilon.$$

□

Now, in order to prove that each variationally Henstock function  $f : [0, 1] \rightarrow X$  is also variationally  $\mathcal{H}$ -integrable, we shall follow the lines of the proof of [27, Theorem 6], with  $E = [0, 1]$ .

Another preliminary result is needed, concerning *interior* Henstock partitions. Since the technique is elementary and quite similar to [27, Lemma 3], we do not present any proof.

**Definition 6.3.** Let  $\delta : [0, 1] \rightarrow \mathbb{R}^+$  be any gauge in  $[0, 1]$ , and let  $P := \{(t_1, I_1), (t_2, I_2), \dots, (t_K, I_K)\}$  be any  $\delta$ -fine Henstock partition of  $[0, 1]$ .  $P$  is said to be an *interior* Henstock partition if  $t_k \in \text{int}(I_k)$  for all  $k$ , except when  $I_k$  contains 0 or 1, in which case  $t_k \in \text{int}(I_k)$  or  $t_k \in I_k \cap \{0, 1\}$ .

The Lemma needed is the following.

**Lemma 6.4.** *Let  $\delta : [0, 1] \rightarrow \mathbb{R}^+$  be any gauge in  $[0, 1]$ , and let  $P := \{(t_1, I_1), (t_2, I_2), \dots, (t_K, I_K)\}$  be any  $\delta$ -fine Henstock partition of  $[0, 1]$ , where the tags  $t_1, \dots, t_K$  are all distinct. Then, for each  $\varepsilon > 0$  there exists a  $\delta$ -fine interior Henstock partition of  $[0, 1]$ ,  $P' := \{(t_1, I'_1), (t_2, I'_2), \dots, (t_K, I'_K)\}$  such that*

$$\sum_{k=1}^K \|f(t_k)\| \|I_k - I'_k\| < \varepsilon.$$

A modified version of the last Lemma will be used later, for variationally Henstock integrable functions.

**Lemma 6.5.** *Let  $f : [0, 1] \rightarrow X$  be any variationally Henstock integrable mapping, and denote by  $\Phi$  its primitive, i.e.  $\Phi(I) = \int_I f$ , for all intervals  $I$ . Suppose that  $\delta : [0, 1] \rightarrow \mathbb{R}^+$  is any gauge in  $[0, 1]$ , and  $P := \{(t_1, I_1), (t_2, I_2), \dots, (t_K, I_K)\}$  is any  $\delta$ -fine Henstock partition of  $[0, 1]$ , whose tags  $t_1, \dots, t_K$  are all distinct. Then, for each  $\varepsilon > 0$  there exists a  $\delta$ -fine interior Henstock partition of  $[0, 1]$ ,  $P' := \{(t_1, I'_1), (t_2, I'_2), \dots, (t_K, I'_K)\}$  such that*

$$\sum_{k=1}^K \|f(t_k)\| \|I_k - I'_k\| < \varepsilon,$$

and

$$\sum_{k=1}^K \|\Phi(I_k) - \Phi(I'_k)\| \leq \varepsilon.$$

*Proof.* Since  $f$  is variationally Henstock integrable, the function  $t \mapsto \Phi([0, t])$  is continuous with respect to the norm topology of  $X$ .  $\square$

We are now ready to give the announced result.

**Theorem 6.6.** *Let  $f : [0, 1] \rightarrow X$  be any variationally Henstock integrable mapping. Then it is also variationally  $\mathcal{H}$ -integrable.*

*Proof.* First of all, we observe that  $f$  is strongly measurable, and therefore strongly Riemann measurable. Fix  $\varepsilon > 0$ . Then there exists a sequence of pairwise disjoint closed sets  $(F_n)_n$  in  $[0, 1]$  and a decreasing sequence  $(\delta_n)_n$  in  $\mathbb{R}^+$  tending to 0, such that the set  $N := \bigcap_n ([0, 1] \setminus F_n)$  has Lebesgue measure 0, and moreover such that for every integer  $n$

$$\sum_{k=1}^K \|f(t_k) - f(t'_k)\| |I_k| \leq \frac{\varepsilon}{2^n}$$

holds, as soon as  $(I_k)_{k=1}^K$  is any non-overlapping family of subintervals with  $\max_k |I_k| < \delta_n$  and the points  $t_k, t'_k$  are taken in  $F_n \cap I_k$ . Now, choose any bounded gauge  $\delta_0$ , corresponding to  $\varepsilon$  in the definition of variational Henstock integral of  $f$ , and set  $\delta(t) = \theta_n(t)$ , when  $t \in F_n$  for some index  $n$ , and  $\delta(t) = \delta_0$  if  $t \in N$ , where

$$\theta_n(t) = \min\left\{\delta_n, \frac{1}{2} \max_{F_n \ni \tau \rightarrow t} \{\delta_0(\tau)\}\right\}$$

$\delta$  is measurable, as proved in [27, Theorem 6]. We shall prove now that the gauge  $\delta/2$  can be chosen in correspondence with  $\varepsilon$  in the notion of variational integrability of  $f$ . To this aim, fix any Henstock partition  $\Pi := \{(t_1, I_1), \dots, (t_K, I_K)\}$

subordinated to  $\delta/2$ . Without loss of generality, we can assume that all tags  $t_k$  are distinct. Indeed, if a tag  $t$  is common to two intervals  $I, J$  of  $\Pi$ , then

$$\left\| f(t)|I| - \int_I f \right\| + \left\| f(t)|J| - \int_J f \right\| \leq 2 \max \left\{ \left\| f(t)|I| - \int_I f \right\|, \left\| f(t)|J| - \int_J f \right\| \right\}$$

and therefore the sum

$$\sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\|$$

is dominated by twice the analogous sum evaluated on a (possibly partial) partition with distinct tags.

Thanks to Lemma 6.5, there exists an *interior* Henstock partition  $\Pi' := \{(t_k, J_k), k = 1, \dots, K\}$  subordinated to  $\delta/2$  and such that

$$(5) \quad \sum_{k=1}^K \|f(t_k)\| |I_k| - |J_k| < \varepsilon, \quad \sum_{k=1}^K \left\| \int_{I_k} f - \int_{J_k} f \right\| \leq \varepsilon.$$

Now, we shall suitably modify the tags of  $\Pi'$ ; fix  $k$  and consider the tag  $t_k$ . If  $t_k \in F_n$  for some  $n$  and  $\limsup_{F_n \ni s \rightarrow t_k} \delta_0(s) \geq \delta_0(t_k)$ , then we pick  $t'_k$  in the set  $\text{int}(I_k) \cap F_n$  in such a way that  $\delta_0(t'_k) > \delta(t_k)$ . This is possible since then we have  $\limsup_{F_n \ni s \rightarrow t_k} \delta_0(s) \geq 2\delta(t_k)$ . If  $t_k \in F_n$  for some  $n$  and  $\limsup_{F_n \ni s \rightarrow t_k} \delta_0(s) < \delta_0(t_k)$  or if  $t_k \in N$ , then we set  $t'_k = t_k$ . From this it follows that the partition  $\Pi'' := \{(t'_k, I_k) : k = 1, \dots, K\}$  is an interior Henstock partition subordinated to  $\delta_0$ . Summarizing, we have

$$\begin{aligned} \sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\| &\leq \sum_k \|f(t_k)|I_k| - f(t_k)|J_k|\| + \sum_k \|f(t_k)|J_k| - f(t'_k)|J_k|\| + \\ &+ \sum_k \left\| f(t'_k)|J_k| - \int_{J_k} f \right\| + \sum_k \left\| \int_{I_k} f - \int_{J_k} f \right\|. \end{aligned}$$

Now,

$$\sum_k \|f(t_k)|I_k| - f(t_k)|J_k|\| + \sum_k \left\| \int_{I_k} f - \int_{J_k} f \right\| \leq 2\varepsilon$$

thanks to (5), and

$$\sum_k \left\| f(t'_k)|J_k| - \int_{J_k} f \right\| \leq \varepsilon$$

because  $\Pi''$  is  $\delta_0$ -fine. Finally, thanks to the strong Riemann measurability,

$$\sum_k \|f(t_k)|J_k| - f(t'_k)|J_k|\| = \sum_{t_k \in N^c} \|f(t_k)|J_k| - f(t'_k)|J_k|\| \leq \sum_n \frac{\varepsilon}{2^n} \leq \varepsilon,$$

and so

$$\sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\| \leq 4\varepsilon$$

which concludes the proof.  $\square$

Observe also that Theorem 6.6 can be extended immediately to multifunctions thanks to Rådström embedding Theorem and so also Theorem 5.2 can be extended to variationally  $\mathcal{H}$  integrable multifunction  $\Gamma$ .

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